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THE POINCARÉ–CHETAYEV EQUATIONS AND FLEXIBLE MULTIBODY SYSTEMS†

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Problems of the dynamics of flexible multibody systems (FMBSs) and its relation to the fundamental system of equations obtained by Poincaré about 100 years ago [1] are considered. These equations, called the Poincaré–Chetayev equations, are now well known as the basis of the Lagrange reduction theory. By extending these equations to the case of the motion of a Cosserat medium it is shown that in the dynamics of FMBS it is possible to use two principal systems of equations. It is proved that a generalized Newton–Euler model of FMBSs in projections onto floating axes and the partial differential equations of the non-linear, geometrically exact theory within the Galilean approach comprise the Poincaré–Chetayev equations.

Interest in the dynamic modelling of flexible multibody systems (FMBSs) [2–10] is due to its applications in the dynamics of systems of two types: rapid light industrial manipulators and large space structures. Depending on which relations are used to describe the link strains, two approaches can employed to model such systems.

The first of these, termed "the floating frame approach" [5–10], is often limited to the region of linear elasticity, since, within its framework, the link strains are regarded as modal perturbations of the principal motions of certain mobile structures. In this approach, the most effective algorithms for simulation and control are based on the New–Euler model [5–9], which is opposed by Lagrangian models [10]. Various methods have been proposed for deriving the Newton–Euler model of FMBSs. Initially, Euler and Newton's laws, together with the method of projections, were used for the derivation [5]. As an alternative, it was suggested [6] that Lagrange's equations in quasi-coordinates should be used. The application of non-holonomic velocities made it possible to use Hamel's equations [7]. The same model can be obtained from the virtual energy principle [8] and using Euler's Lagrange's concepts of the description of motion [11]. For floating frames (systems of coordinates), the Newton and Euler formulation of dynamics has many advantages compared with Lagrangian dynamics. Above all it requires less computational time since, within its framework, it is possible to describe the dynamics of individual links and to connect these descriptions using a recursive kinematic chain model. Furthermore, the dynamic quantities which occur in such a description allow of a simple physical interpretation, which cannot be said of the Lagragian approach, in which the same quantities occur in the kinetic energy in the form of complex expressions. This advantage is crucial when an attempt is being made to add to the consideration certain non-linear effects such as, for example, the effect of dynamic stiffening [12]. Finally, owing to its recursive nature, this model can also be used when fast direct [13] and inverse [14] $o(n)$ algorithms are being employed, where n is the number of links.

Another theory, the Galilean theory, was first developed for large space structures allowing of finite strains and low stresses [2–4]. The following main observation relates to this theory: if elastic displacements are of the same order of magnitude as rigid body displacements, the problem of their separation becomes artificial, and, to separate such motions, too many additional assumptions are required. Thus, in this theory, the systems of coordinates for the link strains are Galilean, and for the distribution of links they are global. For the most part, this approach has been based on Reissner's beam

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theory [15], in which each structural cross-section remains rigid (a Cosserat medium [16]). The configuration space of such a system may be identified with $SO(3) \times R^3$ using a map that fits points from this space to the material line of a beam. A numerical procedure that has been used [2–4] to solve the problem of the rotational motion in weak form rested largely not on one but on a collection of parameters from the corresponding space, as is the case in the Lagrangian approach. The method is based on reducing the order using a three-dimensional finite element method and on a Newmark implicit one-step integration scheme. The standard Newmark "predictor–corrector" scheme has to be modified since the curvature of SO(3) space prevents standard linear operations in vector spaces. Since this obstacle is overcome by projection of the dynamics onto the *initial* configuration, the non-linear Cauchy problem is replaced, using Newton's procedure, by a finite sequence of linear problems. To determine the linear dynamics in tangent space, accurate internal linearization in weak form is achieved by differentiation in absolute space. Since no approximations are made before carrying out spatial and temporal discretizations, such an approach is termed "geometrically accurate".

The aim of the present paper is to show that these equations, or, more accurately, the equations of the generalized Newton–Euler model of FMBSs in the associated system of coordinates, and the equations of the geometrically accurate model of a multibeam system in a Galilean system of coordinates can be represented directly in the form of the set of equations discovered by Poincaré [1] and improved by Chetayev [17, 18]. These equations, known today as the Poincaré–Chetayev equations [19–21] or Euler–Poincaré equations [22], represent a generalization of Lagrange's equations on an optimal commutative Lie group [23, 24]. In the dynamics of FMBSs, unlike standard structural dynamics, elastic elements of the mechanism not only undergo strains but also perform motions as a rigid whole. This is a peculiar feature of Poincaré's results.

In Section 1 we introduce the notation and the basic concepts. In Section 2, following with the construction of the Cosserat brothers [24], the Poincaré–Chetayev equations are extend to the case of a continuum. In Section 3 these equations are applied to an important FMBS – an open-loop flexible manipulator.

1. NOTATION AND BASIC DEFINITIONS

We will briefly recall the definitions of the following tools of differential geometry (cf. [23], Appendix 2) that will be used below.

Let G be an n-dimensional Lie group of transformations $R^3 \mapsto R^3$ with a unit e. The motion of a material system in configuration space G is defined by the map g: $t \in R^+ \mapsto g(t) \in G$. In R^3 space, where the motion is taking place, a set of transformed configurations $\Sigma(t) = g(t)\Sigma_0 \in R^3$, $t \in R^+$ is defined, where $\Sigma_0 \in R^3$ is the initial configuration of the system, also called the "material space". The Lie algebra of group G is denoted by $\mathfrak{g}$ and is defined as the space T_eG tangent to G at e, endowed with a Lie bracket $[\,,]$. We will introduce on G an internal product $\langle\,,\rangle$ and define $\mathfrak{g}^*$ as the vector space of 1-forms on G.

Let L_g and R_g be the left and right translations on G, and let L_{g*} and R_{g*} be the tangent space maps induced by these translations. The translations L_g and R_g, by means of the internal product, induce cotangent maps L_g^* and R_g^* reciprocal to the maps L_{g*} and R_{g*} respectively. Differentiation of group automorphism Ad_g: $h \in G \mapsto R_{g^{-1}}(L_g(h))$ at the point $h = e$ gives the action map Ad_{g*} of group G on $\mathfrak{g}$. Then the differentiation of Ad_{g*} with respect to g at $g = e$ defines the adjoint map $\mathrm{ad}_{(\cdot)*}$ of the map $\mathfrak{g} \mapsto \mathfrak{g}$. By means of its duality, the map $\mathrm{ad}_{(\cdot)*}$ defines the co-adjoint $\mathrm{ad}^*_{(\cdot)*}$: $\mathfrak{g} \mapsto \mathfrak{g}$. The left-invariant (right-invariant) field on G comprises a vector field that is invariant in relation to L_{g*} (R_{g*}).

The vector space of the left-invariant (right-invariant) vector fields on group G, endowed with a Poisson vector field bracket $[\,,]$, realizes a different definition of space $\mathfrak{g}$. On account of its duality, the vector space of the left-invariant (right-invariant) 1-forms on group G realizes a different definition of the space $\mathfrak{g}^*$. The action space $\mathfrak{g} \cong (T_e, G, [\,,])$ is identical to the space of infinitesimal transformations applied to the reference configuration Σ_0 ('right' or 'material' transformations) or to the actual configuration $\Sigma(t)$ ('left' or 'spatial' transformations). Alternatively, the space of left-invariant (right-invariant) fields realizes a different definition of material (spatial) infinitesimal transformations.

To write out the dynamic equations of the material system in terms of infinitesimal material (spatial) transformations is to describe the dynamics in a "material (spatial) approach". Subsequent calculations are conducted "in coordinates", as in the original papers by Poincaré and Chetayev. Agreement concerning the summation over repeated indices is assumed everywhere, apart from the cases specified at the end of Section 2. Finally, we will sometimes use the notation $f(x_1 \ldots, x_k) = f(x_l)(l = 1, \ldots, k)$ for any function f, the vector x and the integer k.

2. THE POINCARÉ–CHETAYEV EQUATIONS FOR A COSSERAT MEDIUM

A Cosserat medium comprises a continuum of microbodies, for example, beam cross-sections, the transverse rigid material lines of a shell, or grains in a micropolar continuum [25], and the spatial configurations of the medium can therefore be described by the action of an n-dimensional group G (in a typical case – SE(3) or SO(3)) on an elementary rigid microbody at each point of the submanifold D of the reference configuration of the medium (reference lines for a beam or reference membranes for a shell). Unlike to the finite-dimensional case studied by Poincaré [1], transformations from the group are parameterized not only time but also by material coordinates X^I ($I = 1, \ldots, p, p \leq 3$) in the space D. We will denote the space of parameters (care must be taken, since these parameters are not group parameters) by $P = R^+ \times D$, where R^+ is the time axis. We will denote in terms of x an arbitrarily chosen point P with the coordinates

$$(x^i)_{i=0,\ldots,p} = (t, X^I)_{I=1,\ldots,p} = (t, X)$$

In order to write the Poincaré–Chetayev equations for a Cosserate medium, Hamilton's principle will be applied to a field of Lagragians of the form

$$\mathcal{L}: P \mapsto \bigwedge^p (T^*D), \quad x \mapsto \mathcal{L}(q(x), \eta_i)dV, \quad i = 0, \ldots, p \tag{2.1}$$

Here, $\mathcal{L}$ is the Lagrangian density in the space D, $T^*D - i$ is the cotangent bundle to the space D, $\wedge$ is the external product, dV is the volume of the p-form in space D, and $q(x)$ is the vector of group parameters of the actual transformation at the point X, applied to the microbody. Then, let $\eta_i(x)$ be the actual infinitesimal transformation allowed by the body, i.e. the transformation of the translation along the ith coordinate line P passing through the point X that is examined the basis of space $\mathfrak{g}$.

Below we will concentrate primarily on the material approach since it is of greatest interest from the viewpoint of mechanics. In this approach, two interpretations are given to the quantities $\eta_i(x)$, depending on the adopted definition of the space $\mathfrak{g}$. If this space is defined as $(T_eG, [\,,])$, then

$$\eta_i(x) = L_{g(x)^{-1}*}(\partial_{x^i} g(x)) = \eta_i^\alpha(x) e_\alpha \tag{2.2}$$

Here and below e_α is the basis of infinitesimal material transformations, i.e. transformations acting on material particles; unless otherwise stated, $i = 0, \ldots, p$; $I = 1, \ldots, p$; $\alpha, \beta, \gamma = 1, \ldots, n$. On the other hand, if the Lie algebra is defined as the space of left-invariant fields furnished with a Poisson bracket, then the quantities η_i^α are defined by the relations

$$\eta_i(x) = \partial_{x^i} g(x) = \eta_i^\alpha(x) X_{\alpha, g(x)} \tag{2.3}$$

where (X_α: $g \mapsto X_{\alpha,g} = L_{g*}(e_\alpha)$, $g \in G$) is the basis of the left-invariant vectors on group G, and here the base point on G corresponds to the basis index. In fact, simple analysis of the differences of expressions (2.2) from expressions (2.3) indicates that the set of these vector fields on P realizes a unique vector field of 1-forms on P with values in the corresponding group of the Lie algebra η: $P \mapsto \mathfrak{g} \otimes T^*P$ [24], where $\otimes$ denotes a tensor product. For example, if $(T_eG, [\,,])$ is adopted as a definition of the Lie algebra, then this vector fields, which has the form

$$\eta(x) = (L_{g^{-1}*}(\partial_{x^i} g))(x) dx^i = \eta_i^\alpha(x) e_\alpha \otimes dx^i \tag{2.4}$$

will define the infinitesimal transformation applied from the left to $g(x)$, accomplished by a shift from an arbitrary point x to a point $x + dx$ in space P. On the other hand, adopting left-invariant vector fields as the definition of the Lie algebra, this field of 1-forms can be represented in the form of the relation

$$\eta(x) = (\partial_{x^i} g)(x) dx^i = \eta_i^\alpha(x) X_{\alpha, g(x)} \otimes dx^i \tag{2.5}$$

which delivers the replacement of $g(x)$ in the field of the left-invariant basis covering group G on transition from x to $x + dx$ on P.

In order to apply Hamilton's principle to a Lagrangian density of the form (2.1), it is necessary to derive a formula which plays a key role in the variation calculus on non-commutative Lie groups. This relation is a consequence of the fact that the variation δ is achieved at fixed time and fixed material parameters. To substantiate this result, we note first of all that the variation of the function f from $C^\infty(G)$ at the point $g(x)$ has the form

$$\delta f(g(x)) = \delta g(x) f = \Omega^\alpha(x) X_{\alpha, g(x)} f \tag{2.6}$$

where $\Omega^\alpha(x)$ are the components of virtual infinitesimal transformations in the basis of left-invariant fields. On the other hand, the derivative of any function f with respect to the parameter τ of the space–time curve γ: $\tau \in R \mapsto \gamma(\tau) \in P$ passing through point x has the form

$$\frac{df(g(x))}{d\tau} = \frac{dg(x)}{d\tau} f = (\eta_i^\alpha(x) X_{\alpha, g(x)} \otimes dx^i)(f, \xi_x) = \eta_i^\alpha(x)\xi^i(x) X_{\alpha, g(x)} f \tag{2.7}$$

where $\xi_x = \xi^i(x)\partial_{x^i}$ is the tangent vector to curve γ at the point x. Moreover, we note that, in particular, when the condition $\tau = x^j$ is satisfied, the relation $\xi_x = \partial_{x^j}$ is satisfied, and expression (2.7) takes the form

$$\partial_{x^j} f(g(x)) = (\partial_{x^j} g(x)) \cdot f = (\eta_i^\alpha(x) X_{\alpha, g(x)} \otimes dx^i)(f, \partial_{x^j}) = \eta_j^\alpha(x) X_{\alpha, g(x)} f \tag{2.8}$$

and therefore, for a variation with a fixed time and a fixed position of the medium

$$\frac{d}{d\tau}\delta f - \delta\frac{df}{d\tau} = 0 \tag{2.9}$$

for any function f of class $C^\infty(G)$ and for any curve γ passing through the point x.

Substituting expressions (2.6) and (2.7) into equality (2.9) with g equivalent to x, we obtain

$$\frac{d}{d\tau}\delta f - \delta\frac{df}{d\tau} = \eta_i^\alpha \xi^i X_{\alpha, g}(\Omega^\beta X_{\beta, g} f) - \Omega^\beta X_{\beta, g}(\eta_i^\alpha \xi^i X_{\alpha, g} f) = 0$$

Since this relation must be satisfied for any curve γ, i.e. for any set ξ^i, the condition that the parameters be fixed can be rewritten as

$$\begin{aligned} &\eta_i^\alpha (X_{\alpha, g}\Omega^\beta)(X_{\beta, g} f) + \eta_i^\alpha \Omega^\beta X_{\alpha, g}(X_{\beta, g} f) - \\ &- \Omega^\beta (X_{\beta, g}\eta_i^\alpha)(X_{\alpha, g} f) - \Omega^\beta \eta_i^\alpha X_{\beta, g}(X_{\alpha, g} f) = 0, \ \forall f \in C^\infty(G) \end{aligned} \tag{2.10}$$

Using relations (2.6) and (2.8) we can find on the left-hand side of equality (2.10) both the terms $\Omega^\beta(X_{\beta, g}\eta_i^\alpha) = \delta\eta_i^\alpha$ and $\eta_j^\alpha(X_{\alpha, g}\Omega^\beta) = \partial_{x^j}\Omega^\beta$ and the Poisson bracket of the left-invariant basis vectors $[X_\alpha, X_\beta] = c_{\alpha\beta}^\gamma X_\gamma$ with the structure constants $c_{\alpha\beta}^\gamma$ of the Lie algebra $\mathfrak{g}$. As a result, we have the following relations

$$\delta\eta_i^\alpha = \partial_{x^i}\Omega^\alpha + c_{\beta\gamma}^\alpha \eta_i^\beta \Omega^\gamma \tag{2.11}$$

Equations (2.11) generalize the corresponding one-parameter formula [1].

In exactly the same way, the condition that the parameters be fixed, expressed in the right-invariant basis, can be written as

$$\delta\mu_i^\alpha = \partial_{x^i}\Omega^\alpha - c_{\beta\gamma}^\alpha \mu_i^\beta \Omega^\gamma \tag{2.12}$$

where the structure constants in the right-invariant basis Z_α: $g \mapsto Z_{\alpha, g} = R_{g*}(e_\alpha)$, $g \in G$ are opposite to those in the left-invariant basis [22].

Before using Hamilton's principle, it is necessary to model the fields of external forces applied to the medium. Geometrically, the resultant of the actual forces applied to the microbody at the point X with configuration $g(x)$ is an element of the space $\mathfrak{g}^*$. Therefore, we will define two fields of external forms with the values in the dual space of the Lie algebra – the field of external forces applied within the medium, which has the form

$$\bar{F}: R^+ \times D^0 \mapsto \mathfrak{g}^* \otimes \bigwedge^p (T^* D^0), \quad (t, X) \mapsto \bar{F} = \bar{\mathcal{F}}(x, g(x))\omega_{g(x)}^\alpha \otimes dV \tag{2.13}$$

and the field of actual external forces applied at the boundary of the medium, which has the form

$$\tilde{F}\colon R^{+} \times \partial D \mapsto \mathfrak{g}^{*} \otimes \bigwedge^{p}(T^{*}\partial D), \quad (t, X) \mapsto \tilde{F} = \tilde{\mathcal{F}}(x, g(x))\omega^{\alpha}_{g(x)} \otimes dS \tag{2.14}$$

where $\omega^{\alpha}\colon \mathfrak{g} \mapsto \omega^{\alpha}_{g}$, $g \in G$ is the basis of the left-invariant 1-forms, dual to the basis X_{α}, and dS is the surface $(p-1)$-form at the boundary ∂D.

We will henceforth use the following notation of the integrals

$$\int(\cdot)dt = \int_{t_1}^{t_2}(\cdot)dt, \quad \int(\cdot)dV = \int_{D}(\cdot)dV, \quad \int(\cdot)dS = \int_{\partial D}(\cdot)dS$$

We will now write the modified Hamilton's principle

$$\delta A = \delta\iint \mathcal{L}(\eta^{\alpha}_{i}, q^{\beta})dVdt = \int \delta W_{\text{ext}}dt, \ \forall\delta g; \quad \delta W_{\text{ext}} = \int \bar{\mathcal{F}}_{\alpha}\Omega^{\alpha}dV + \int \tilde{\mathcal{F}}_{\alpha}\Omega^{\alpha}dS \tag{2.15}$$

where A is the action of the Lagrangian, and δW_{ext} is the virtual work of the external forces applied to the medium.

Proceeding to the variation of action, we have

$$\delta A = \delta\iint \mathcal{L}(\eta^{\alpha}_{i}, q^{\beta})dVdt = I_1 + I_2$$

$$I_1 = \iint \frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}\delta\eta^{\alpha}_{i}dVdt, \quad I_2 = \frac{\partial \mathcal{L}}{\partial q^{\beta}}\delta q^{\beta}dVdt$$

and, by virtue of constraints (2.11),

$$I_1 = I_{11} + I_{12}, \quad I_{11} = \iint \frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}\partial_{x^i}\Omega^{\alpha}dVdt, \quad I_{12} = \iint \frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}c^{\alpha}_{\beta\gamma}\eta^{\beta}_{i}\Omega^{\gamma}dVdt \tag{2.16}$$

Since

$$I_{11} = \iint\left[\partial_t\left(\frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{0}}\Omega^{\alpha}\right) + \partial_{x^I}\left(\frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{I}}\Omega^{\alpha}\right)\right]dVdt$$

then the relations

$$\frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}\partial_{x^i}\Omega^{\alpha} = \partial_{x^i}\left(\frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}\Omega^{\alpha}\right) - \partial_{x^i}\left(\frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{i}}\right)\Omega^{\alpha} \tag{2.17}$$

integration by parts with respect to time and the fact that the variation $\delta g(x) = \Omega^{\alpha}X_{\alpha, g(x)}$ vanishes at the ends of the time interval, yield

$$I_{11} = \iint \frac{\partial \mathcal{L}}{\partial \eta^{\alpha}_{I}}\Omega^{\alpha}N_I dSdt \tag{2.18}$$

where N_I is the Ith component of the unit normal to the boundary ∂D.

By virtue of relation (2.6)

$$\frac{\partial \mathcal{L}}{\partial q^{\beta}}\delta q^{\beta} = \frac{\partial \mathcal{L}}{\partial q^{\beta}}\Omega^{\alpha}X_{\alpha, g}(q^{\beta})$$

and the following equality holds

$$I_2 = \iint \frac{\partial \mathcal{L}}{\partial q^{\beta}}\Omega^{\alpha}X_{\alpha, g}(q^{\beta})dVdt \tag{2.19}$$

Then, by virtue of relations (2.17)–(2.19), the modified Hamilton principle (2.15) takes the form

$$0 = \int\Big(\int\Big(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} N_I - \tilde{\mathcal{F}}_\alpha\Big)\Omega^\alpha dS +$$
$$+ \int\Big(-\partial_{x^i}\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} + c_{\beta\alpha}^\gamma \eta_i^\beta \frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} + \frac{\partial \mathcal{L}}{\partial q_\beta} X_{\alpha,g}(q^\beta) - \bar{\mathcal{F}}_\alpha\Big)\Omega^\alpha dV\Big)dt$$

Since the equation holds for any variation, i.e. for any independent Ω^α, we have the following assertion.

Assertion. In the space $\mathfrak{g}^*$, identical to the space of left-invariant 1-forms, the field equation and the relations at the boundary have the form

$$\Big(\frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} - c_{\beta\alpha}^\gamma \eta_i^\beta(x)\frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} - \frac{\partial \mathcal{L}}{\partial q^\beta} X_{\alpha,g(x)}(q^\beta) - \bar{\mathcal{F}}(x, g(x))\Big)\omega_{g(x)}^\alpha = 0,\ \forall x \in R^+ \times D^0 \tag{2.20}$$

$$\Big(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} N_I(x) - \tilde{\mathcal{F}}_\alpha(x, g(x))\Big)\omega_{g(x)}^\alpha = 0,\ \forall x \in R^+ \times \partial D \tag{2.21}$$

Note that, to calculate term (2.19) there is no need to introduce charts of parameters in space G. In fact, using the basis of infinitesimal material transformations (e_α) of space $(T_eG, [\,,])$, we have

$$\frac{\partial \mathcal{L}}{\partial q_\beta} X_{\alpha,g}(q_\beta) = \Big[\frac{d}{d\varepsilon}\mathcal{L}(\eta(x), L_{g(x)}(\exp(\varepsilon e_\alpha)))\Big]_{\varepsilon=0} \tag{2.22}$$

where $\exp\colon (T_eG, [\,,]) \mapsto G$ is the natural map of the Lie algebra into the group [23] and where the Lagrangian density is now a function of the transformations, i.e. $\mathcal{L} = \mathcal{L}(\eta, g)$.

The terms (2.22) are responsible for the defect of the symmetry of the Lagrange function in the material approach (cf. Remark 4 below). Now, if it is required to write the Poincaré–Chetayev equations in the spatial approach, the field of 1-forms with the values in Lie Algebra (2.5) is replaced by

$$\mu(x) = (\partial x^I g)(x)dx^I = \mu_i^\alpha(x)Z_{\alpha,g(x)} \otimes dx^i$$

The application of the same calculation process to the spatial Lagrangian density $\mathcal{L}(\mu, q)dV$, taking into account relation (2.12) instead of (2.11) yields

$$\Big(\frac{\partial}{\partial x^i}\frac{\partial \mathcal{L}}{\partial \mu_i^\alpha} + c_{\beta\alpha}^\lambda \mu_i^\beta(x)\frac{\partial \mathcal{L}}{\partial \mu_i^\lambda} - \frac{\partial \mathcal{L}}{\partial q^\beta} Z_{\alpha,g(x)}(q^\beta) - \bar{\mathscr{C}}_\alpha(x, g(x))\Big)\lambda_{g(x)}^\alpha = 0,\ \forall x \in R^+ \times D^0 \tag{2.23}$$

$$\Big(\frac{\partial \mathcal{L}}{\partial \mu_I^\alpha} N_I(x) - \tilde{\mathscr{C}}_\alpha(x, g(x))\Big)\lambda_{g(x)}^\alpha = 0,\ \forall x \in R^+ \times \partial D \tag{2.24}$$

where $\lambda^\alpha\colon g \mapsto \lambda_g^\alpha, g \in G$ is the basis of right-invariant 1-forms, dual to Z_α, and $\bar{\mathscr{C}}_\alpha$ and $\tilde{\mathscr{C}}_\alpha$ are components of the fields of forms of the external and internal boundary forces in this basis, which, in the general case, depend on x and $g(x)$.

Finally, the terms

$$\frac{\partial \mathcal{L}}{\partial q_\beta} Z_{\alpha,g(x)}(q_\beta) = \Big[\frac{d}{d\varepsilon}\mathcal{L}(\mu(x), R_{g(x)}(\exp(\varepsilon e_\alpha)))\Big]_{\varepsilon=0} \tag{2.25}$$

are responsible for the defect of the symmetry of the Lagrange function in the spatial approach. Here, $\exp(\varepsilon e_\alpha)$ is now applied on the left of $g(x)$, so that e_α is the basis of infinitesimal spatial transformations, i.e. transformations applied to the points of space.

We will single out in relations (2.20) and (2.23) the components of the co-adjoint map $\mathrm{ad}_{(\cdot)}^*$: $\mathfrak{g}^* \mapsto \mathfrak{g}^*$ in the dual to the left- and right-invariant bases respectively [24]. We will apply to each of the approaches (2.20), (2.21) and (2.23), (2.24) the cotangent maps $L_{g(x)}^*$ and $R_{g(x)}^*$. In the basis

$f^\alpha = L^*_{g(x)}(\omega^\alpha_{g(x)}) = R^*_{g(x)}(\lambda^\alpha_{g(x)})$ on $\mathfrak{g}$, where the algebra $\mathfrak{g}$ is now defined as $(T_eG, [\,,])$, the dynamics is defined by the equations

$$\frac{\partial}{\partial x^i}\frac{\partial \mathcal{L}}{\partial \eta_i} - \mathrm{ad}^*_{\eta_i}\frac{\partial \mathcal{L}}{\partial \eta_i} - X_{g(x)}(\mathcal{L}) = \bar{\mathcal{F}}, \quad \forall x \in R^+ \times D^0 \tag{2.26}$$

$$\frac{\partial \mathcal{L}}{\partial \eta_I}N_I = \tilde{\mathcal{F}}, \quad \forall x \in R^+ \times \partial D \tag{2.27}$$

where the following notation is introduced

$$\begin{gathered} X_{g(x)}(\mathcal{L}) = \left[\frac{d}{d\varepsilon}\mathcal{L}(\eta(x), L_{g(x)}(\exp(\varepsilon e_\alpha)))\right]_{\varepsilon=0} f^\alpha \\ \frac{\partial \mathcal{L}}{\partial \eta_i} = \frac{\partial \mathcal{L}}{\partial \eta_i^\alpha}f^\alpha, \quad \frac{\partial \mathcal{L}}{\partial \eta_I} = \frac{\partial \mathcal{L}}{\partial \eta_I^\alpha}f^\alpha, \quad \bar{\mathcal{F}} = \bar{\mathcal{F}}_\alpha f^\alpha, \quad \tilde{\mathcal{F}} = \tilde{\mathcal{F}}_\alpha f^\alpha \end{gathered} \tag{2.28}$$

Similarly, in the spatial approach

$$\frac{\partial}{\partial x^i}\frac{\partial \mathcal{L}}{\partial \mu_i} + \mathrm{ad}^*_{\mu_i}\frac{\partial \mathcal{L}}{\partial \eta_i} - Z_{g(x)}(\mathcal{L}) = \bar{\mathcal{E}}, \quad \forall x \in R^+ \times D^0 \tag{2.29}$$

$$\frac{\partial \mathcal{L}}{\partial \eta_I}N_I(x) = \tilde{\mathcal{E}}, \quad \forall x \in R^+ \times \partial D \tag{2.30}$$

where the following notation is introduced

$$\begin{gathered} Z_{g(x)}\mathcal{L} = \left[\frac{d}{d\varepsilon}\mathcal{L}(\mu(x), R_{g(x)}(\exp(\varepsilon e_\alpha)))\right]_{\varepsilon=0} f^\alpha \\ \frac{\partial \mathcal{L}}{\partial \mu_i} = \frac{\partial \mathcal{L}}{\partial \mu_i^\alpha}f^\alpha, \quad \frac{\partial \mathcal{L}}{\partial \mu_I} = \frac{\partial \mathcal{L}}{\partial \mu_I^\alpha}f^\alpha, \quad \bar{\mathcal{E}} = \bar{\mathcal{E}}_\alpha f^\alpha, \quad \tilde{\mathcal{E}} = \tilde{\mathcal{E}}_\alpha f^\alpha \end{gathered} \tag{2.31}$$

Note that, in relations (2.26) and (2.27), the basis f_α is a dual basis to infinitesimal material transformations, while in (2.29) and (2.30) this is the dual basis to infinitesimal spatial transformations.

Remark 1. Even if the system is continuous and in certain sense infinite-dimensional, the group used in the construction given above is finite but parameterized by the material manifold D (we will say that the group is measured on D [24]). This distinguishes the system under examination radically from systems examined in fluid mechanics, where the configuration space is an infinite-dimensional group [26]. In fact, the configuration space of Cosserat medium is the set $\mathcal{C} = \{g: D \mapsto G\}$.

Remark 2. Comparing relations (2.20), (2.21) and (2.26), (2.27) ((2.23), (2.24) and (2.29), (2.30) respectively), we note that the Poincaré–Chetayev equations are written in projections onto the same left-invariant (right-invariant) dual basis as the dual basis of the infinitesimal material (spatial) transformations. Consideration of the right- and left-invariant fields for determining the Lie algebra corresponds in fact to the well-known mobile-basis method proposed by Cartan [27] for investigating certain problems of integrability. This viewpoint also has it analogue in the action space of the group, where, as is well known, the material formulation in the components is similar to the spatial formulation in a mobile basis connected to body.

Remark 3. When the parameter space P reduces to the time axis R^+, the partial differential equations (2.26), (2.27) and (2.29), (2.30) degenerate into classical Poincaré–Chetayev ordinary differential equations [1].

Assertion. Finite-dimensional Poincaré–Chetayev equations in the material formulation have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_0} - \mathrm{ad}^*_{\eta_0}\frac{\partial L}{\partial \eta_0} - X_{g(t)}(L) = F, \ \forall t \in R^+$$
$$X_{g(t)}(L) = [L(\eta_0, L_{g(t)}(\exp(\varepsilon e_\alpha)))]_{\varepsilon = 0} f^\alpha, \quad \frac{\partial L}{\partial \eta_0} = \frac{\partial L}{\partial \eta_0^\alpha} f^\alpha \tag{2.32}$$

Finite-dimensional Poincaré–Chetayev equations in the spatial formulation have a form similar to (2.32):

$$\frac{d}{dt}\frac{\partial L}{\partial \mu_0} + \mathrm{ad}^*_{\mu_0}\frac{\partial L}{\partial \eta_0} - Z_{g(t)}(L) = E, \ \forall t \in R^+$$
$$Z_{g(t)}(L) = [L(\mu_0, R_{g(t)}(\exp(\varepsilon e_\alpha)))]_{\varepsilon = 0} f^\alpha, \quad \frac{\partial L}{\partial \mu_0} = \frac{\partial L}{\partial \mu_0^\alpha} f^\alpha \tag{2.33}$$

where $L = L(\eta_0, g)$ ($L = L(\mu_0, g)$) is the Lagrange function of the system in the material (spatial) formulation, and $F = F_\alpha(t, g)f^\alpha$ ($E = E_\alpha(t, g)f^\alpha$) is the 1-form of the external material (spatial) forces applied to the system.

Remark 4. The above equations were generalized and connected with certain systems of equations of analytical dynamics by Rumyantsev [20] in the case when the Lie algebra of invariant fields is replaced by an arbitrary closed system of infinitesimal linear operators X_α. In this case, the Poincaré–Chetayev equations, in which the structure constants of the Lie algebra are replaced by variable coefficients $c^\lambda_{\beta\alpha}$, remain valid.

Remark 5. These equations are particularly intersecting when the Lagrangian and the density of external forces are independent of the configuration $g(x)$. This case has been widely studied [23, 22]. It relates to the Lagrange reduction theory. In the given context, if the Lagrange function of the system and, in the case examined here, the external forces also are invariant under left transformations (for a rigid body) or right transformations (for an incompressible fluid), then, once expressed in its Lie algebra, it becomes independent of the configuration. The resultant dynamic equations have the form (2.26), (2.27) or (2.29), (2.30), in which terms of the symmetry defect (2.22) (correspondingly (2.25)) no longer occur, and, furthermore, components of forces (2.13) and (2.14) (and their spatial analogues) are independent of the configuration. In modern terminology, the dynamics reduces to dynamics on a Lie algebra $\mathfrak{g}$ of the group of symmetries of the system. Thus, these equations comprise partial differential equations in terms of the velocities η_0 or μ_0 only (for the Euler formulation). Consequently, they can first be integrated over time, and, only at the second step, owing to the equations

$$\partial_t g((x)) = L_{g(x)*}(\eta_0(x)) \quad \text{or} \quad \partial_t g(x) = R_{g(x)*}(\mu_0(x))$$

is it possible to restore the motion of the medium.

Remark 6. If the group G is commutative, then

$$X_\alpha = Z_\alpha, \quad [X_\alpha, X_\beta] = [Z_\alpha, Z_\beta] = 0$$

Consequently, the field of local bases (X_α) is derived from the set q^α of coordinate functions from the class $C^\infty(G)$ and satisfies the relations

$$X_\alpha = \partial_{g^\alpha}, \quad \omega^\alpha = dq^\alpha$$

As a result we have $\eta^\alpha = \dot{q}^\alpha$, and the velocities turn out to the integrable, i.e. holonomic. In the latter case, the Poincaré–Chetayev equations (2.32) and (2.33) degenerate into a unique system of Lagrange equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha}\right) dq^\alpha = 0$$

3. APPLICATIONS TO FLEXIBLE MULTIBODY SYSTEMS

In order to demonstrate the application of the Poincaré–Chetayev equations to flexible multibody systems, we will examine the special case of a flexible manipulator. We will show how these equations enable us to write geometrically accurate equations in a Galilean reference frame and generalized Newton–Euler equations in a floating reference frame. Extensions to other forms of topology of multibody systems are straightforward.

The partial differential equations for a flexible manipulator in a Galilean system of reference frame

Kinematic Principles of a Body. In this approach, each body in the system is modelled in accordance with Reissner's non-linear beam theory [15]. Below we will examine the case of a rectilinear beam of initial length L of constant cross-section. Let Σ_0 be the initial beam configuration. Its points are material particles whose positions X are equal to $X^I E_I$ in the material coordinate system (O, E_1, E_2, E_3), attached to ω_0, such that E_1 is the material axis of the beam. In the spatial system of coordinates (O, e_1, e_2, e_3), taken practically to coincide with the material system of coordinates, the actual position of the particle X is specified in the form

$$x(X, t) = x^i(t, X^I)e_i$$

In Reissner's theory, the beam cross-sections are assumed to be rigid, and therefore they realize microbodies of a one-dimensional Cosserate medium. Thus, according to the general construction proposed in Section 2, the parameter space here is $P = R^+ \times [0, L]$. The actual three-dimensional field of beam positions is determined by the action of the transformation φ_r: $\Sigma_0 \mapsto R^3$ parameterized by time and defined as

$$\begin{aligned} x(t, X) &= \varphi_t(X) = X^1 e_1 + d(t, X^1) + R(t, X^1)(X^2 E_2 + X^3 E_3) \\ \forall X &= (X^1, X^2, X^3) \in \Sigma_0 \end{aligned} \tag{3.1}$$

where $d(t, X^1)$ is the actual translation applied to the centres of mass of the cross-section X^1, and $R(t, X^1)$ is the rotation of this same cross-section.

Transformation (3.1) can be rewritten in the homogeneous formalism as

$$\left\| \begin{matrix} x(t, X) \\ 1 \end{matrix} \right\| = \left\| \begin{matrix} R(t, X^1) & d(t, X^1)X^1 e_1 \\ 0 & 1 \end{matrix} \right\| \left\| \begin{matrix} r \\ 1 \end{matrix} \right\| = g(t, X^1) \left\| \begin{matrix} r \\ 1 \end{matrix} \right\| \tag{3.2}$$

where the homogeneous 4×4 matrices g realize an SE(3) group of Euclidean displacements in three-dimensional space R^3. Thus, the configuration space of the beam is realized as

$$\mathscr{C} = \{g: [0, L] \mapsto \mathrm{SE}(3)\}$$

(see Remark 1). Furthermore, the field g acts on a subset of the set

$$\Sigma_0: \mathscr{V} = \{r = X^2 E_2 + X^3 E_3/(X^I E_I \in \Sigma_0)\}$$

which plays the role of a "typical" microbody. The Lie algebra of the SE(3) group, denoted by se(3), is identical here to the space of twists R^6, endowed with a product denoted by an asterisk such that

$$\eta * \eta' = \left\| \begin{matrix} \Omega \\ V \end{matrix} \right\| * \left\| \begin{matrix} \Omega' \\ V' \end{matrix} \right\| = \left\| \begin{matrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \end{matrix} \right\|, \quad \forall \eta, \eta' \in R^6 \tag{3.3}$$

In accordance with the material approach, the algebra se(3) is identical here to infinitesimal material rigid displacements with the basis

$$\left\| \left\| \begin{matrix} E_I \\ 0 \end{matrix} \right\|_{I = 1, 2, 3}, \left\| \begin{matrix} 0 \\ E_I \end{matrix} \right\|_{I = 1, 2, 3} \right\| = (e_\alpha)_{\alpha = 1, \ldots, 6} \tag{3.4}$$

and a field of 1-forms with values in Lie algebra (2.4)

$$\eta: R^+ \times [0, L] \mapsto se(3) \otimes T^*(R^+ \times [0, L])$$

$$\eta(t, X^1) = \eta_0 \otimes dt + \eta_1 \otimes dX^1 = \begin{Vmatrix} \Omega(t, X^1) \\ V(t, X^1) \end{Vmatrix} \otimes dt + \begin{Vmatrix} K(t, X^1) \\ \Gamma(t, X^1) \end{Vmatrix} \otimes dX^1 \tag{3.5}$$

where η_0 and η_1 are the twists associated with isomorphism between se(3) and R^6 in $g^{-1}\partial_t g$ and $g^{-1}\partial_{X^1} g$ respectively. It is intuitively clear that $\eta_0(t, X^1)$ is a material infinitesimal transformation allowing the transition from mobile axes connected to the cross-section X^1 at the instant t to mobile axes at the instant $t + dt$. At the same time, $\eta_1(t, X^1)$ is an infinitesimal transformation making it possible at a fixed time t to switch from mobile axes connected to the cross-section X^1 to mobile axes connected to the cross-section $X^1 + dX^1$. The dual space of the Lie algebra se(3)* is identical to the space of wrenches, isomorphic to R^6, the duality product of twists and wrenches reduces to the duality product in R^6. If

$$\eta = \begin{Vmatrix} \Omega \\ V \end{Vmatrix}, \quad \lambda = \begin{Vmatrix} \Lambda \\ W \end{Vmatrix}$$

are arbitrary vectors from se(3) and its dual space respectively, then the co-adjoint action of η on λ is defined as [22]

$$\mathrm{ad}^*_\eta(\lambda) = \begin{Vmatrix} \Lambda \times \Omega + W \times V \\ W \times \Omega \end{Vmatrix} \tag{3.6}$$

Measure of strains of a body. We will now define the measure of the strains of a body adopted in Reissner's theory. There are two strain fields of the beam [2]:

1. The vector field of material strains, which will be denoted by ε and defined as

$$(t, X^1) \in R^+ \times [0, L] \mapsto \varepsilon(t, X^1) = R^T \partial_{X^1} \varphi_t(X^1, 0, 0) - E_1 = \Gamma(t, X_1) - E_1 \tag{3.7}$$

where the vector component along E_1 is measure of the stretching of the beam, while the other two relate to the transverse shearing.

2. The field of material curvature

$$(t, X^1) \in R^+ \times [0, L] \mapsto R^T \partial_{X^1} R = \hat{K}(t, X_1) \tag{3.8}$$

the material tensor of which has already been introduced by the second of equalities (3.5) through the field of the pseudovector K linked with $\hat{K}$ by the natural isomorphism so(3) $\mapsto R^3$ (this concept will be used systematically below).

The Lagrangian of a body. We will now write the velocity field as

$$\partial_t \varphi_t(X) = \partial_t d + (\partial_t R) r$$

Assuming that the reference line of the beam passes through the centres of mass of the cross-sections, the kinetic energy will be expressed in the form

$$2T = \int_{\Sigma_0} (\partial_t \varphi_t)^2 dm = \rho A \int_0^L [(\partial_t d)^T \partial_t d + ((\partial_t R) r)^T (\partial_t R) r dX^1]$$

where A is the cross-section area. Then, introducing the vector η_0: $\eta_0^T = (\Omega^T, V^T)$, the expression for the kinetic energy can be rewritten as

$$T = \frac{1}{2} \int_0^L \rho \eta_0^T \mathbf{J} \eta_0 dX^1 = \int_0^L \mathcal{T}(\eta_0) dX^1, \quad \mathbf{J} = \begin{Vmatrix} J & 0 \\ 0 & J \end{Vmatrix} \tag{3.9}$$

where $\mathcal{T}$ is the kinetic energy density and ρJ is the material tensor of inertia of the cross-section

$$\rho J = \rho\iint_A \hat{r}^T\hat{r}dX^2dX^3 = \rho I_p\mathcal{E}_1 + \rho I_a\mathcal{E}_2 + \rho I_a\mathcal{E}_3; \quad \mathcal{E}_k = E_k \times E_k, \quad k = 1, 2, 3$$

Here, I_a and I_p are the axial and polar moments of inertia of a typical cross-section.

Considering the case of an elastic material with small strains, and introducing the vector η_1: $\eta_1^T = (K^T, (\Gamma - E_1)^T)$, the strain energy can be approximated by the quadratic potential of measures of strains (3.7) and (3.8)

$$T = \frac{1}{2}\int_0^L \eta_1^T \mathbf{H}\eta_1 dX^1 = \int_0^L \mathcal{U}(\eta_1)dX^1, \quad \mathbf{H} = \begin{Vmatrix} H_r & 0 \\ 0 & H_d \end{Vmatrix} \tag{3.10}$$

where $\mathcal{U}$ is the energy density of the strains, H_d and H_r are the reduced Hooke's tensors for the beam in a material coordinate system

$$H_d = EA\mathcal{E}_1 + GA\mathcal{E}_2 + GA\mathcal{E}_3, \quad H_r = GI_p\mathcal{E}_1 + EI_a\mathcal{E}_2 + EI_a\mathcal{E}_3$$

E is Young's modulus, and G is the tension modulus,s Finally, the Lagrangian takes the form

$$\mathcal{L}(\eta_0, \eta_1) = \mathcal{T}(\eta_0) - \mathcal{U}(\eta_1) \tag{3.11}$$

and no longer depends on the beam configuration, i.e. is left-invariant. This is unsurprising since the left-invariance of the elastic potential corresponds to the principle of independence of the choice of coordinate system, while the right-invariance relates to the isotrophy of the elastic properties of the medium. On the other hand, the left-invariance of the kinetic energy corresponds to the isotrophy of space, while the right-invariance corresponds to the isotrophy of inertial properties of the material.

The Poincaré–Chetayev equations for a body. **Field equations.** A link is subject to the action of external left-invariant (follower) forces and moments

$$(t, X^1) \mapsto \bar{F} = \bar{\mathcal{F}}dX^1 = \begin{Vmatrix} \bar{m}(t, X^1) \\ \bar{n}(t, X^1) \end{Vmatrix} dX^1 \tag{3.12}$$

and of left-invariant (follower) forces and moments applied at its extreme points

$$X^1 = 0: t \mapsto \tilde{F}_-(t) = \begin{Vmatrix} \tilde{M}_-(t) \\ \tilde{N}_-(t) \end{Vmatrix}; \quad X^1 = L: t \mapsto \tilde{F}_+(t) = \begin{Vmatrix} \tilde{M}_+(t) \\ \tilde{N}_+(t) \end{Vmatrix} \tag{3.13}$$

Examining Eqs (2.26) and (2.27) with $x^0 = t$ and $x^1 = X^1$, it is possible to derive the Poincaré–Chetayev equations for a free one-dimensional Cosserat medium with a Lagrangian density of the form (3.11)

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{T}}{\partial \eta_0} - \mathrm{ad}^*_{\eta_0}\frac{\partial \mathcal{T}}{\partial \eta_0} - \frac{\partial}{\partial X^1}\frac{\partial \mathcal{U}}{\partial \eta_1} + \mathrm{ad}^*_{\eta_1}\frac{\partial \mathcal{U}}{\partial \eta_1} = \bar{\mathcal{F}} \tag{3.14}$$

The application of the co-adjoint maps $\mathrm{ad}^*_{\eta_0}$ and $\mathrm{ad}^*_{\eta_1}$, defined by relations (3.6), to the kinetic energy $\mathcal{T}$ and the potential energy $\mathcal{U}$ for any $(t, X^1) \in R^+ \times]0, L[$ accordingly yields

$$\begin{Vmatrix} \rho J(\partial_t\Omega + \Omega \times J\Omega) \\ \rho A(\partial_t V + \Omega \times V) \end{Vmatrix} = \begin{Vmatrix} \partial_{X^1}M + K \times M + (R^T\partial_{X^1}\phi) \times N + \bar{m} \\ \partial_{X^1}N + K \times N + \bar{n} \end{Vmatrix} \tag{3.15}$$

Here, we have introduced the position field of the reference line of the beam $\varphi_t(X^1, 0, 0) = \phi(t, X^1)$, and also the force and moment of internal forces applied on the cross-section X^1

$$\left\| \begin{matrix} H_r K \\ H_d \varepsilon \end{matrix} \right\| = \left\| \begin{matrix} M \\ N \end{matrix} \right\| = \left(\left(\frac{\partial \mathcal{L}}{\partial K} \right)^T, \left(\frac{\partial \mathcal{L}}{\partial \Gamma} \right)^T \right) \tag{3.16}$$

where M is the moment of the field of internal forces applied to the cross-section and evaluated at the centre of mass of this section, and N is the resultant force.

Equations at the boundary. The application of the general equations (2.27) with $N_1(0) = -1$ and $N_1(L) = 1$ yields

$$\frac{\partial \mathcal{L}}{\partial \eta_1}(t, 0) = -\tilde{F}_-(t), \quad \frac{\partial \mathcal{L}}{\partial \eta_1}(t, L) = \tilde{F}_+(t), \quad t \in R^+$$

Finally, by virtue of equality (3.15)

$$\left\| \begin{matrix} M(t,0) \\ N(t,0) \end{matrix} \right\| = \left\| \begin{matrix} M_-(t) \\ N_-(t) \end{matrix} \right\|, \quad \left\| \begin{matrix} M(t,L) \\ N(t,L) \end{matrix} \right\| = \left\| \begin{matrix} M_-(t) \\ N_-(t) \end{matrix} \right\|, \quad t \in R^+ \tag{3.17}$$

Equations (3.15)–(3.17) are Reissner's partial differential equations [15]. They can be interpreted as tensor equations in material space or, alternatively, as equations in terms of the components in a field of mobile axes $(R(t, X^1)E_1, R(t, X^1)E_2, R(t, X^1)E_3)$. To integrate these equations, they must be closed by means of Hooke's law, the equations $\partial_t R = R\hat{\Omega}$ and $\partial_t \phi = RV$, which make it possible to recover the change in configuration, and definitions of the strain measures.

Remark. Reissner's equations were written [15] in the spatial formulation and were derived by applying the Poincaré–Chetayev equations (2.29) and (2.30) to the spatial Lagrangian, which also depends on the configuration. In this case, to obtain a correct result, it is necessary to calculate the symmetry defect $Z_{g(x)}(L)$.

The partial differential equations of a flexible manipulator. Consider the special case of the motion of a manipulator in zero gravity. The manipulator consists of p links denoted (from base to end-point) by $B_0, B_1 \ldots, B_p$. The base B_0 is assumed to be rigid and fixed, while the remaining links are modelled by Reissner beams. The links are connected by cylindrical hinges denoted (from base to end-hinge) by a_1, $a_2 \ldots, a_p$. Torques τ_j are applied to the hinges, acting on the corresponding links a_j (here and everywhere below, unless specified otherwise, $j = 1, 2, \ldots, p$); it is assumed that these torques are concentrated at points. All the notation adopted is retained in the case of a single link, apart from a discarded index. The vectors a_j are three-dimensional. We will define their material analogue as

$$A_j(t) = R_j^T(t, L_j)a_j(t) = R_{j+1}^T(t, 0)a_j(t)$$

Using these material vectors, we will define the operator $A_j^{\perp}$ projecting any vector V onto the space perpendicular to A_j:

$$A_j^{\perp}: R^3 \mapsto R^2, \quad V \mapsto A_j^{\perp} V = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right\| (V - (A_j V)V)$$

In this notation, the dynamics of the manipulator is described by a system of equations including: *the field equations*

$$\left\| \begin{matrix} \rho_j(J_j \partial_t \Omega_j + \Omega_j \times J_j \Omega_j) \\ \rho_j A_j(\partial_t V_j + \Omega_j \times V_j) \end{matrix} \right\| = \left\| \begin{matrix} \partial_{X^1} M_j + K_j \times M_j + (R_j^T \partial_{X^1} \phi_j) \times N_j \\ \partial_{X^1} N_j + K_j \times N_j \end{matrix} \right\|, \quad X^1 \in]0, L_j[\tag{3.18}$$

the equations for reconstructing the links configurations, written in accordance with the arbitrary reference position of the manipulator,

$$\partial_t R_j(t, X^1) = R_j(t, X^1)\hat{\Omega}_j(t, X^1), \quad \partial_t \phi_j(t, X^1) = R_j(t, X^1) V_j(t, X^1), \quad X^1 \in [0, L_j] \tag{3.19}$$

the equations for reconstructing the rotations at the hinges

$$^jR_{j+1}(t) = R_j^T(t, L_j) R_{j+1}(t, 0), \quad ^0R_{j+1}(t) = R_1(t, 0), \quad j = 1, \ldots, p-1 \tag{3.20}$$

and the boundary conditions for the links

$$\begin{gathered} N_j(t, 0) = -\tilde{N}_j(t), \ N_j(t, L_j) = -\tilde{N}_{j+1}(t), \ M_j(t, 0) = \tilde{M}_j(t), \ M_j(t, L_j) = -\tilde{M}_{j+1}(t) \\ N_p(t, 0) = -\tilde{N}_p, \ N_p(t, L_p) = 0, \ M_p(t, 0) = -\tilde{M}_p(t), \ M_p(t, L_p) = 0 \ \ j = 1, \ldots, p-1 \end{gathered} \tag{3.21}$$

where the end-point of the manipulator is assumed to be free, $\tilde{N}_j$ and $\tilde{M}_j$ are the force and torque acting on the link B_j from the B_{j-1} side, and $\tilde{N}_j$ and $A_j^{\perp}\tilde{M}_j$ are vectors of the Lagrange multipliers that were introduced in order for the following constraint conditions to be satisfied

$$\begin{gathered} V_j(t, L_j) = {}^jR_{j+1}(t) V_{j+1}(t, 0), \quad A_j^{\perp}(t)\Omega_j(t, L_j) = A_j^{\perp}(t)^j R_{j+1}(t)\Omega_{j+1}(t, 0) \\ V_1(t, 0) = 0, \quad A_1^{\perp}\Omega_1(t, 0) = 0; \quad j = 1, \ldots, p-1 \end{gathered} \tag{3.22}$$

The equations of a flexible manipulator in the floating reference frame

The kinematic participles of a link. In the approach based on the use of a floating reference frame each link of the manipulator is regard as a three-dimensional elastic body undergoing small strains superimposed on the finite motions as a whole. We will examine a typical free link in the reference configuration $\Sigma_0 \subset R^3$. We will provide the link Σ_0 with the material system of coordinates (O, E_1, E_2, E_3). The actual configuration of the body will be denoted by $\Sigma(t)$. It is embedded in the geometric space R^3 provided with a spatial system of coordinates (O, e_1, e_2, e_3). The material and spatial systems of coordinates will be examined together.

In this approach the transformation φ_t, mapping Σ_0 onto $\Sigma(t)$, may be written as a composition of two transformations. The first of these is the pure strain mapping Σ_0 onto $\Sigma_0(t)$; it is denoted by φ_t^e. The second, denoted by φ_t^r, comprises the displacement as a rigid whole, converting $\Sigma_0(t)$ into $\Sigma(t)$. In this way we have a sequence of transformations

$$\varphi_t = \varphi_t^r \circ \varphi_t^e \colon \Sigma_0 \mapsto \Sigma_0(t) \mapsto \Sigma(t)$$

transferring the point mass X to a point in space x by the following rule

$$X(t, X) = \varphi_t(X)\varphi_t^r(\varphi_t^e(X)) \tag{3.23}$$

As earlier, the transformation corresponding to the displacement of points of the body as a rigid whole is written in the form

$$\varphi_t^r(X') = d_0(t) + R(t)X' \tag{3.24}$$

where X' is a point form $\Sigma_0(t)$, and d_0 is the displacement of a reference point of the body. For elastic transformations it can be written as

$$\varphi_t^e(X) = X' = X + d(t, X) \tag{3.25}$$

where d is the field of displacements of material origin, mapping the position of a particle in the reference configuration onto its image owing to pure strain.

As earlier, the set of all transformations as a rigid whole realizes a Lie group SE(3), acting in the given case on the deformed configuration $\Sigma_0(t)$. Here, φ_t^e is a point of diff(Σ_0) – the space of diffeomorphisms of the space R^3 into itself restricted to Σ_0. Moreover, the floating reference frame is identified with a mobile system of coordinates – an image of the material coordinate system – by a component delivering the transformation as a rigid whole.

With the composition of maps (3.23), the configuration space of the body forms the group

$$G = \mathrm{SE}(3) \times \mathrm{diff}(\Sigma_0)$$

As regards the space diff(Σ_0), we will replace it with final group

$$D(t, X) = \sum_{\alpha=1}^{m} \Phi_\alpha(X)q^\alpha(t) = \Phi_\alpha(X)q^\alpha(t), \quad \forall X \in \Sigma_0 \tag{3.26}$$

called the modal reduction, where Φ_α represents the natural modes of the body under certain boundary conditions (the modal indices are denoted by Greek letters, and the indices in space R^3 by Latin letters). They comprise material vectors, i.e. $\Phi_\alpha = \Phi_\alpha^K E_K$.

Such modal decomposition presupposes above all that the body is subject to small strains. Under these conditions, the group of diffeomorphisms diff(Σ_0) is parameterized by a vector of modal coordinates $q = (q^1, q_2^2, \dots, q_m^m)^T$ and, from geometric considerations, is replaced by the linear space R^m, i.e. a commutative Lie group. Then the Lie group $G = \mathrm{SE}(3) \times R^m$ realizes the configuration space of the elastic body, and any two transformations g and g' from this group G are composed as

$$g \circ g' = \left\| \begin{matrix} \left\| \begin{matrix} R & d \\ 0 & 1 \end{matrix} \right\| \\ q \end{matrix} \right\| \circ \left\| \begin{matrix} \left\| \begin{matrix} R' & d' \\ 0 & 1 \end{matrix} \right\| \\ q' \end{matrix} \right\| = \left\| \begin{matrix} \left\| \begin{matrix} RR' & Rd' + d \\ 0 & 1 \end{matrix} \right\| \\ q + q' \end{matrix} \right\| \tag{3.27}$$

The Lie algebra $\mathfrak{g}$ of group G is realized s se(3) $\times R^m$, and, owing to the natural isomorphism assigning the space R^6 to the space se(3), also as space R^{6+m} provided with a product, denoted by an asterisk, such that

$$\eta * \eta' = \left\| \begin{matrix} \Omega \\ V \\ \dot{q} \end{matrix} \right\| * \left\| \begin{matrix} \Omega' \\ V' \\ \dot{q}' \end{matrix} \right\| = \left\| \begin{matrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \\ (\dot{q} + \dot{q}') - (\dot{q} + \dot{q}') \end{matrix} \right\| = \left\| \begin{matrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \\ 0 \end{matrix} \right\| \tag{3.28}$$

where $\eta, \eta' \in R^{6+m}$, and $\dot{q} = (\dot{q}^1, \dot{q}_2^2, \dots, \dot{q}_m^m)^T$ is the vector of modal velocities (the dot denotes a time derivative). Furthermore, in accordance with the material formulation of the problem, the space R^{6+m} is identified with the space of infinitesimal material transformations of the basis

$$(e_\alpha)_{\alpha = 1, \dots, 6+m} = \left\| \left\| \begin{matrix} E_I \\ 0 \\ 0 \end{matrix} \right\|_{I = 1,2,3}, \left\| \begin{matrix} 0 \\ E_I \\ 0 \end{matrix} \right\|_{I = 1,2,3}, \left\| \begin{matrix} 0 \\ 0 \\ \partial_{q^\alpha} \end{matrix} \right\|_{\alpha = 1, \dots, m} \right\|$$

Also, the 1-form with values in Lie algebra (2.4) is reduced to

$$\eta(t) = \eta_0(t) \otimes dt, \quad \eta_0(t) = \left\| \begin{matrix} \Omega \\ V \\ \dot{q} \end{matrix} \right\| \tag{3.29}$$

where η_0 is a vector from R^{6+m}, associated with $L_{g^{-1}*}(\dot{g})$ by means of the isomorphism between se(3) and R^6, i.e. satisfying the relations $\hat{\Omega} = R^T\dot{R}$ and $V = R^T d_0$.

The dual space to the algebra se(3) $\times R^m$, that is, se(3)* $\times R^m$, is again the space R^{6+m}. In it there are six first components – components of the wrench in the material system of coordinates, and m final components – components of the generalized modal forces. As regards the duality product, it is reduced to a duality product in R^{6+m} space. Finally, the co-adjoint action of any vector $\xi \in \mathfrak{g}$ on any vector λ from its dual space, by virtue of relation (3.29), has the form

$$\mathrm{ad}^*_\xi(\lambda) = \mathrm{ad}_{\eta_0}\begin{Vmatrix} \Lambda \\ W \\ \Omega \end{Vmatrix} = \begin{Vmatrix} \Lambda\times\Omega + W\times V \\ W\times\Omega \\ 0 \end{Vmatrix} \tag{3.30}$$

The Lagrangian of a single link Bearing in mind relations (3.24)–(3.26), from equality (3.23) we have

$$\varphi_t(X) = d_0(t) + R(t)(X + \Phi_\alpha(X)q^\alpha(t))$$

and therefore the velocity field has the form

$$\dot\varphi_t(X) = \dot d_0(t) + \dot R(t)(X + \Phi_\alpha(X)q^\alpha(t)) + R\Phi_\alpha(X)\dot q^\alpha(t)$$

Substituting this expression into the expression for kinetic energy of the body, and distinguishing the vector η_0 from the Lie algebra, we find

$$T = \frac{1}{2}\int_{\Sigma_0}\dot\varphi_t^T\dot\varphi_t dm = \frac{1}{2}mV^TV + \frac{1}{2}\Omega^TJ\Omega + \frac{1}{2}m_{\alpha\beta}\dot q^\alpha\dot q^\beta + V^T(\Omega\times ms + a_\alpha\dot q^\alpha) + \Omega^T\beta_\alpha\dot q^\alpha$$

where expressions for the kinetic energy of translation and rotation are written from left to right, followed by the expression for the kinetic energy of the strains, and, finally, mixed terms. Here, the following tensors are introduced (integration is carried out over points to link Σ_0)

$$m_{\alpha\beta} = \int\Phi_\alpha^T\Phi_\beta dm,\quad a_\alpha = \int\Phi_\alpha dm,\quad \alpha_\beta = \int X\times\Phi_\beta dm,\quad \lambda_{\alpha\beta} = \int\Phi_\alpha\times\Phi_\beta dm$$
$$m = \int dm,\quad ms = \int(X + \Phi_\alpha q^\alpha)dm,\quad \beta_\nu = \alpha_\nu + \lambda_{\gamma\nu}q^\gamma$$
$$J = \int(X + \Phi_\alpha q^\alpha)^T(X + \Phi_\beta q^\beta)dm = J_{rr} + (J_{re,\alpha} + J^T_{er,\alpha})q^\alpha + J_{ee,\alpha\beta}q^\alpha q^\beta$$

Besides the kinetic energy, the potential energy of the strains is specified, defined as the quadratic form of the modal coordinates

$$2U = K_{\alpha\beta}q^\alpha q^\beta = q^TKq$$

where K is the matrix of modal stiffness. Finally, the Lagrangian of a free link takes in se(3) $\times R^m$ the reduced form

$$L(\eta_0, q) = \frac{1}{2}\eta_0^T\mathcal{J}\eta_0,\quad -\frac{1}{2}q^TKq,\quad \mathcal{J} = \begin{Vmatrix} J & m\hat s & \beta \\ m\hat s^T & mI & a \\ \beta^T & a^T & M \end{Vmatrix} \tag{3.31}$$

where $I = \mathrm{diag}(1, 1, 1)$, $M = (m_{\alpha\beta})_{\alpha,\beta=1,\dots,m}$ is the matrix of the generalized modal inertia, $\beta = (\beta_1 \dots, \beta_m)$ and $a = (a_1, \dots, a_m)$ are the matrices of material vectors, and $m\hat s$ is a skew-symmetric tensor such that $m\hat s\upsilon = ms\times\upsilon$, $\forall\upsilon\in R^3$.

The Poincaré–Chetayev Equations for a Link The finite-dimensional Poincaré–Chetayev equations (2.32) with Lagragian (3.31), having the form

$$\frac{d}{dt}\frac{\partial T}{\partial\eta_0} + \mathrm{ad}^*_{\eta_0}\frac{\partial T}{\partial\eta_0} - X_g(L) = 0 \tag{3.32}$$

after some reduction can represented as

$$\frac{d}{dt}\frac{\partial T}{\partial\eta_0} - \mathrm{ad}^*_{\eta_0}\frac{\partial T}{\partial\eta_0} = \mathcal{J}\begin{Vmatrix} \dot\Omega \\ \dot V \\ \ddot q \end{Vmatrix} + \begin{Vmatrix} \dot J\Omega + \Omega\times(J\Omega + \beta\dot q) + ms\times(\Omega\times V) \\ \Omega\times(mV + 2s\dot q + \Omega ms) \\ \dot\beta^T\Omega \end{Vmatrix}$$

Here, we have taken into account the relations

$$m\dot{s} = a_\alpha \dot{q}^\alpha = aq, \quad \beta\dot{q} = \beta_\alpha \dot{q}^\alpha = 0$$
$$V \times (\Omega \times ms) - \Omega \times (V \times ms) = (V \times \Omega) \times ms = ms \times (\Omega \times V)$$

Now, since Lagragian (3.31) depends on the configuration only in terms of strains, expression (2.32) for the defect of the symmetry is written in the form

$$X_g(L) = \left(0, 0, \left(\frac{d}{d\varepsilon}L(\eta, q + \varepsilon\partial_q)\right)^T_{\varepsilon = 0}\right)^T = (0, 0, (\partial_q L)^T)$$

$$\frac{\partial L}{\partial q_\alpha} = \frac{1}{2}\Omega^T(J_{re,\alpha} + J_{er,\alpha})\Omega + \frac{1}{2}q^\beta\Omega^T(J_{ee,\alpha\beta} + J_{ee,\beta\alpha})\Omega + \dot{q}^\beta\Lambda^T_{\alpha\beta}\Omega + V^T(\Omega \times a_\alpha) - K_{\alpha\beta}q^\beta$$

Noting that $\lambda_{\gamma\alpha} = -\lambda_{\alpha\gamma}$ and that for any 3×3 matrix A

$$-\Omega^T((A^T + A)\Omega) = -(A\Omega)^T\Omega - \Omega^T(A\Omega) = -2\Omega^T A\Omega$$

the defect of the symmetry can be represented in the form of an m-dimensional column vector

$$\partial_q L = \Omega^T J_{er,\alpha}\Omega - q^\beta\dot{\Omega}^T J_{ee,\alpha\beta}\Omega + 2\dot{q}^\beta\lambda^T_{\alpha\beta}\Omega + K_{\alpha\beta}q^\beta$$

Here, the dynamics of a free elastic body in space

$$\mathfrak{g} = \mathrm{se}(3) \times R^m \cong R^{6+m}$$

taking relation (3.32) into account, is defined by the equations

$$\mathbf{J}\left\|\begin{matrix}\dot{\Omega} \\ \gamma \\ \ddot{q}\end{matrix}\right\| + \left\|\begin{matrix}\Omega \times J\Omega + 2J_{re,\alpha}\Omega\dot{q}^\alpha + 2J_{ee,\alpha\beta}\Omega q^\beta\dot{q}^\alpha \\ 2\Omega \times a\dot{q} + \Omega \times (\Omega \times ms) \\ \Omega^T J_{er,\alpha}\Omega - q^\beta\Omega^T J_{ee,\alpha\beta}\Omega + 2\dot{q}^\beta\lambda^T_{\alpha\beta}\Omega + K_{\alpha\beta}q^\beta\end{matrix}\right\| = \left\|\begin{matrix}0 \\ 0 \\ 0\end{matrix}\right\| \tag{3.33}$$

where the material acceleration of the reference point $\gamma = \dot{V} + \Omega \times V$ is introduced.

Equation (3.33) are well known to specialists in the area of multibody mechanics in floating reference frames. Here, they correspond to the "generalized Newton–Euler model" for an elastic body [5–10]. They can be interpreted as tensor equations in material space or as equations in terms of components in the floating reference frame $(R(t)E_1, R(t)E_2, R(t)E_3)$. To integrate them it is necessary to close Eqs (3.33) with the equation $\dot{R} = R\hat{\Omega}$.

The generalized Newton–Euler model for a flexible manipulator We will now consider the same manipulator as in the first part of Section 3. Each link is now modelled using the approach associated with the introduction of floating reference frames attached to hinged points $O_1, O_2, \ldots O_p$ between the bodies, comprising the base of the mechanism; the modal functions of the form are defined by the same points. The wrenches induced by hinges belong to the slave-link type. If all the tensors are given the same indices as the links, the generalized Newton–Euler model for a flexible manipulator will be specified by three systems of equations:

the equations of the dynamics of the links

$$\left\|\begin{matrix}J_{rrj} & J_{rej} \\ J_{rej} & J_{eej}\end{matrix}\right\| \left\|\begin{matrix}\dot{V}_j \\ \ddot{q}_j\end{matrix}\right\| + \left\|\begin{matrix}C_j \\ c_j\end{matrix}\right\| = \left\|\begin{matrix}F_j - {}^jT_{j+1}F_{j+1} \\ -\Phi_j^{Tj}R_{j+1}F_{j+1}\end{matrix}\right\|, \quad j = 0, \ldots, p \tag{3.34}$$

the model of velocities

$$V_j = {}^jT_{j-1}V_{j-1} + {}^jR_{j-1}\Phi_{j-1}\dot{q}_{j-1} + \dot{q}_{rj}A_j, \quad j = 1, \ldots, p \tag{3.35}$$

the model of accelerations

$$\dot{V}_j = {}^jT_{j-1}\dot{V}_{j-1} + {}^jR_{j-1}\Phi_{j-1}\ddot{q}_{j-1} + H_j, \quad j = 1, \dots, p \tag{3.36}$$

Relation (3.34) was derived from relation (3.33) using the replacement

$$(\dot{V}_j^T, \ddot{q}_j^T) = (\dot{\Omega}_j^T, \gamma_j^T, \ddot{q}_j^T)$$

and F_j is wrench applied from the j-th body to the next body. The $6 \times m$ matrix Φ_j, determining the displacements and the rotations of the form vector of the jth link, is defined at the point O_{j+1}. The 6×6 matrix ${}^jR_{j-1}$ specifies the transformation of the floating reference frame. The 6×6 matrix ${}^jT_{j-1}$ corresponds to the transformation of the screw of the $(j - 1)$th link from intrinsic reference frame to that of the following body. The six-dimensional vector A_j defines the axis of the jth hinge. Finally, the vector H_j determines the Coriolis and centrifugal accelerations arising at the jth hinge.

Equations (3.34)–(3.36) were first written out in [5] and later in [6–9] (see [12–14] for details of their use in the dynamics of flexible multibody systems).

4. CONCLUSIONS

The proposed extension of the Poincaré–Chetayev equations to a Cosserat medium shows how the two principal sets of equations used in the dynamics of flexible manipulators can be described using natural language. The partial differential equations obtained within the Galilean approach form the basis of the geometrically accurate approach in a numerical investigation of flexible multibody systems [2–4]. On the other hand, the use of floating reference frame to describe the link strains makes it possible to identify configuration space with the Cartesian product of the SE(3) group and the space of generalized coordinates describing the strains. From the geometric viewpoint, reduction (trivialization) of the dynamics of the elastic body to smooth layering occurs, in which the commutative subgroup acts as a base manifold (here, the modal space or, more generally, the "space of forms"), while the layers are a non-commutative subgroup (here, SE(3)) [22]. We note, finally, that these equations have numerous applications in multibody dynamics; in particular, they have made it possible to construct $0(n)$ algorithms, where n is the number of links in problems of inverse and direct dynamics of a flexible manipulator in relative coordinates [13, 14]. The modern application of these equations in robotics touches on the study of the motion of systems with many joints. In this case the elastic manifold from the examples considered earlier is replaced by a manifold of joints of the multibody system. The control problem that arises consists of the following: what should the motions at the joints be to ensure, by control means, the possibility of adequate motion on SE(3)? This is the main question in the theory of the motion of animals [28].

REFERENCES

1. POINCARÉ, H., Sur une forme nouvelle des équations de la mécanique. *C. R. Acad. Sci. Paris*, 1901, **132**, 369–371.
2. SIMO, J. C. and VU-QUOC, L., On the dynamics in space of rods undergoing large motions – a geometrically exact approach. *Comput. Methods in Appl. Mech. and Eng.*, 1988, **66**, 125–161.
3. CARDONA, A. and GÉRADIN, M., A beam finite element non-linear theory with finite rotations. *Int. J. Numer. Methods in Eng.*, 1988, **26**, 2403–2438.
4. IBRAHIMBEGOVIC, A. and AL MIKDAD, M., Finite rotations in dynamics of beams and implicit time-stepping schemes. *Int. J. Numer. Methods in Eng.*, 1998, **41**, 781–814.
5. HIGHES, P. C. and SINCARSIN, G. B., Dynamics of elastic multibody chains. Part A. Body motion equations. *J. Dynamics and Stability of Systems*, 1898, **4**, 209–226.
6. MEIROVITCH, L., Hybrid state equations of motion for flexible bodies in terms of quasi-coordinates. *J. Guidance*, 1991, **14**, 1008–1013.
7. BREMER, H., Fast moving flexible robot dynamics. In *Symp. on Robot Control (SYROCO)*, Nates, France, 1997, 45–52.
8. BOYER, F. and COIFFET, P., Generalisation of Newton–Euler model for flexible manipulators. *J. Robotic Systems*, 1996, **13**, 1, 11–24.
9. BOYER, F., Contribution à la modélisation et commande dynamique des robots flexibles. PhD thesis, University Paris VI, Paris, 1994.
10. BOOK, J. W., Recursive Lagrangian dynamics of flexible manipulator arms. *Int. J. Robotic Research*, 1984, **3**, 87–101.
11. GERMAIN, P., *Cours de Mécanique de l'Ecole Polytechnique*, Vol. 1, *Ellipses*. Paris, 1986.
12. BOYER, F., GLADAIS, N. and KHALIL, W., Flexible multibody dynamics based on non-linear Euler–Bernoulli kinematics. *Int. J. Numer. Methods in Eng.*, 2002, **54**, 27–59.
13. D'ELEUTERIO, G. M. T., Dynamics of an elastic multibody chain. Part C. Recursive dynamics. *Dynam. Stab. Syst.*, 1992, **7**, 61–89.

14. BOYER, F. and KHALIL, W., An efficient calculation of flexible manipulator inverse dynamics. *J. Robotic Research, 1998,* **17**, 3, 282–293.
15. REISNER, E., On a one-dimensional large displacement finite strain beam theory. *Stud. Appl. Math.*, 1973, **52**, 87–95.
16. COSSERAT, E. and COSSERAT, F., *Théorie des corps déformables*. Hermann, Paris, 1909.
17. CHETAYEV, N. G., Sur les équations de Poincaré. *C. R. Acad. Sci. Paris*, 1927, **185**, 26, 1577.
18. CHETAYEV, N. G., *Theoretical Mechanics*. Nauka, Moscow, 1987.
19. RUMYANTSEV, V. V., On the Poincaré–Chetayev equations. *Prikl. Mat. Mekh.*, 1994. **58**, 3, 373–386.
20. RUMYANTSEV, V. V., General equations of analytical dynamics. *Prikl. Mat. Mekh.*, 1996, **60**, 899–909.
21. CHEVALLIER, D. P., Dynamics from Euler and Lagrange viewpoints. *In Problems of Investigating the Stability and Stabilization of Motion*, Vol. 2. Vychise. Tsentr. Ross. Akad. Nauk, Moscow, 2000, 40–68.
22. MARSDEN, J. E. and RATIU, T. S., *Introduction to Mechanics and Symmetry*. Springer, Berlin, 1999.
23. ARNOLD, V. I., *Mathematical Methods in Classical Mechanics*. Springer, New York, 1988.
24. POMMARET, J. F., *Partial Differential Equations and Group Theory*. Kluwer, Amsterdam, 1994.
25. NAGHDI, P. M., Finite deformation of elastic rods and shells. In *Proc. IUTAM Symp. on Finite Elasticity*, Lehigh University, 1980. Martinus Nijhoff, Boston, 1982, 47–103.
26. ARNOLD, V. I., Sur la géométrie différentielle des groupes de Lie de dimension infinite et ses applications à l'hydrodynamique des fluides parfaits. *Ann. l'Inst. J. Fourier*, 1966, **16**, 1, 319–361.
27. CARTAN, E., *La Théorie des Groupes Finis et Continus et la Géométrie Différentielle Traitée par la Méthode du Repère Mobile.* Gautier-Villar, Paris, 1937.
28. OSTROWSKI, J. P., Computing reduced equations for robotic systems with constraints and symmetries. *IEEE Transactions on Robotics and Automation*, 1999, **15**, 1, 111–123.

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